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A COLUMN GENERATION TECHNIQUE FOR THE COMPUTATION OF STATIONARY POINTS

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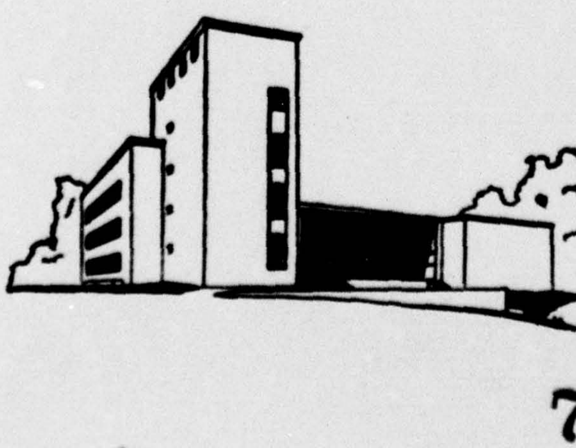

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6 # A COLUMN GENERATION TECHNIQUE FOR THE COMPUTATION OF STATIONARY POINTS

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A COLUMN GENERATION TECHNIQUE FOR THE

COMPUTATION OF STATIONARY POINTS

Jong-Shi Pang

ABSTRACT. In two recent papers, Eaves showed that Lemke's algorithm can be used to compute a stationary point of an affine function over a polyhedral set. This paper proposes an alternative method which is based on parametric principal pivoting. The proposed method involves solving systems of linear equations and parametric linear subprograms over the given polyhedral set. An obvious advantage of the method is that any special structure of the polyhedral set can be exploited profitably in the solution of the subprograms.

1. INTRODUCTION

Given a polyhedral set

$$X = \{x \in R^n \quad : \quad Cx \geqq c\}$$

and an affine mapping

$$F(x) = b + Ax$$

from R^n into R^n, the stationary point problem is to find a vector $u \in X$ such that the condition below is satisfied

(1.1) $$(x - u)^T F(u) \geqq 0 \qquad \text{for all } x \in X \quad .$$

Such a vector u is called a stationary point.

As pointed out by Eaves [3], the stationary point problem is central to the solution of certain quadratic programs, matrix games and economic equilibrium. An important special case of the problem is where the set X is a polyhedral cone. In this case, it has been shown (see [6] e.g.) that the stationary point problem is equivalent to the generalized linear complementarity problem: find $u \in X$ such that

$$F(u) \in X^* \quad \text{and} \quad u^T F(u) = 0$$

where X^* is the dual cone of X, i.e.,

$$X^* = \{y \in R^n \; : \; y^T x \geqq 0 \quad \text{for all } x \in X\} \; .$$

In the reference, Eaves showed that Lemke's algorithm can be used to solve the stationary point problem. His method of analysis can be briefly summarized as follows. First observe that the stationary point problem (1.1) is equivalent to finding vectors u, s and t such that

$$Cu - s = c \quad , \quad b + Au + C^T t = 0$$

$$s, t \geq 0 \quad \text{and} \quad s^T t = 0 .$$

He then adjoins an additional set of constraints $Bx \leq a$ so that the system

$$Cx \geq c \quad , \quad Bx \leq a$$

has a unique solution in the polyhedron X. Next, he applies complementary pivoting to the augmented system

$$\begin{array}{ll} Cu - s \qquad = c & b + Au + C^T t + B^T w = 0 \\ Bu + y - e\,\theta = a & s^T t = y^T w = 0 \\ s \geq 0 \ , \ t \geq 0 \ , \ y \geq 0 \quad , & w \geq 0 \end{array}$$

starting with θ equal to zero and increasing θ to infinity. In a finite number of pivots, the algorithm terminates either on a ray or with a desired stationary point to the given problem. Basically, no specific assumption is needed to operate the algorithm.

In a related paper [4], Eaves describes another way to start Lemke's algorithm for solving the same problem.

Our purpose in the present paper is to propose an alternative approach for solving the stationary point problem. The approach is based on parametric principal pivoting. The ideas involved are briefly sketched as follows. By using the representation of X in terms of its extreme points and rays, it is first shown that the stationary point problem can be converted into an equivalent linear complementarity problem. Under a certain positive semi-definiteness assumption on the matrix A, the resulting linear complementarity problem has a positive semi-definite matrix. Consequently, the parametric principal pivoting algorithm [2, 7] is applicable. The application, however, is crucially dependent on the knowledge of the set of extreme points and rays of X. As the task of generating all these points and rays is practically impossible, it is therefore important to be able to implement the algorithm without the full knowledge of the generators. By means of a column generation technique similar to the one in linear programming, we shall show how the useful components can be generated when they are needed, thereby establishing the applicability of the parametric principal pivoting algorithm for solving the stationary point problem. We shall also discuss a considerably simplified version of this algorithm applicable to the case where the matrix A is positive definite.

2. AN EQUIVALENT LINEAR COMPLEMENTARITY PROBLEM

Since X is polyhedral, there exists a finite set of vectors $\{P_1,\ldots,P_m, Q_1,\ldots,Q_\ell\}$ so that if P and Q denote respectively, the n by m and n by ℓ matrices whose columns are the P_i and Q_j, then we have

$$X = \{x \in R^n : x = P\eta + Q\xi \ : \ \eta, \xi \geqq 0 \quad \text{and} \quad e^T\eta = 1\} \tag{2.1}$$

where e is a vector of ones. Each P_i (Q_j) is an extreme point (ray respectively) of X.

It is obvious that the vector u is a stationary point if and only if it solves the linear program

$$\text{minimize} \quad x^T F(u) \qquad \text{subject to } x \in X .$$

Under the representation (2.1), the latter program is equivalent to the one below

$$\text{minimize} \quad (F(u)^T P)\eta + (F(u)^T Q)\xi \text{ subject to } \eta, \xi \geqq 0 \text{ and } e^T\eta = 1 .$$

By the duality theory of linear programming and by recalling that $F(u) = b + Au$, it follows that $u = P\eta^* + Q\xi^*$ with $\eta^*, \xi^* \geqq 0$ and $e^T\eta^* = 1$ is a stationary point if and only if (η^*, ξ^*) and some suitable λ^+ and λ^- solves the linear complementarity problem below

$$\begin{aligned}
\varphi &= P^Tb + P^TAP\eta + P^TAQ\xi - \lambda^+e + \lambda^-e \geqq 0 & \eta &\geqq 0 \\
\pi &= Q^Tb + Q^TAP\eta + Q^TAQ\xi \geqq 0 & \xi &\geqq 0 \\
\mu^+ &= -1 + e^T\eta \geqq 0 & \lambda^+ &\geqq 0 \\
\mu^- &= 1 - e^T\eta \geqq 0 & \lambda^- &\geqq 0
\end{aligned} \tag{2.2}$$

$$\varphi^T\eta = \pi^T\xi = (\mu^+)^T\lambda^+ = (\mu^-)^T\lambda^- = 0 .$$

In other words, the stationary point problem can always be cast, theoretically, as an ordinary linear complementarity problem of the form

(2.3) $\qquad w = q + Mz \geq 0 \quad , \quad z \geq 0 \quad \text{and} \quad w^T z = 0$

where the vector q and matrix M are given by

(2.4) $$q = \begin{pmatrix} P^T b \\ Q^T b \\ -1 \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} P^T AP & P^T AQ & -e & e \\ Q^T AP & Q^T AQ & 0 & 0 \\ e^T & 0 & 0 & 0 \\ -e^T & 0 & 0 & 0 \end{pmatrix} .$$

It should be pointed out that the derivation of this equivalent linear complementarity problem does not require any assumption on the set X and the matrix A.

3. THE PARAMETRIC PRINCIPAL PIVOTING ALGORITHM

Throughout this paper, we assume that the matrix A is positive semi-definite over the linear subspace spanned by the set X. (This assumption is much weaker than that of a positive semi-definite A.) It is not difficult to show that this assumption is equivalent to the fact that the matrix

$$\begin{pmatrix} P^TAP & P^TAQ \\ Q^TAP & Q^TAQ \end{pmatrix} = \begin{pmatrix} P^T \\ Q^T \end{pmatrix} A\ (P\ Q)$$

is positive semi-definite. Thus, so is the matrix M in (2.4). In particular, the parametric principal pivoting algorithm [2, 7] can be used to solve the linear complementarity problem (2.2). However, this approach is certainly ineffective if it is necessary to know the whole matrices P and Q. To demonstrate how the algorithm can be implemented without the full knowledge of these matrices of generators, we first state a version of the algorithm which operates by updating the constant and parametric columns only.

The Parametric Principal Pivoting Algorithm.

Step 0 (Initialization) Let $J = \emptyset$ and let I be the complement of J.

Step 1 (Computing the basic components) Solve the system of linear equations[1/] for $(\bar{q}_J, \bar{p}_J)$:

(3.1a) $$M_{JJ}(\bar{q}_J\ ,\ \bar{p}_J) = -\ (q_J\ ,\ p_J).$$

1/ If M is a matrix and K and L are index sets, by M_{KL} we mean the submatrix of M whose columns and rows are indexed by K and L respectively. A similar notation is used for vectors.

<u>Step 2</u> (Computing the nonbasic components) Compute

$$(\bar{q}_I, \bar{p}_I) = (q_I, p_I) + M_{IJ}(\bar{q}_J, \bar{p}_J) \tag{3.1b}$$

<u>Step 3</u> (Ratio test) If the vector $\bar{p} = \begin{pmatrix} \bar{p}_I \\ \bar{p}_J \end{pmatrix}$ is nonpositive, terminate with the solution

$$z^* = \begin{pmatrix} z_I^* \\ z_J^* \end{pmatrix} = \begin{pmatrix} 0 \\ \bar{q}_J \end{pmatrix} \tag{3.2}$$

to the linear complementarity problem (2.3). Otherwise determine

$$\theta = \max \{ -\bar{q}_j / \bar{p}_j \quad : \quad \bar{p}_j > 0 \} \tag{3.3}$$

and let k be a maximizing index. If $\theta \leq 0$, terminate with the solution given by (3.2). If $\theta > 0$, continue.

<u>Step 4</u> (Check pivot element) There are two cases

(i) $k \in J$. Solve the system of linear equations for f_J :

$$M_{JJ} f_J = e^k \tag{3.4a}$$

where e^k is a unit vector with a one in component k. Compute

$$f_I = M_{IJ} f_J \quad . \tag{3.4b}$$

(ii) $k \notin J$. Solve the system of linear equations for f_J :

$$M_{JJ} f_J = - M_{Jk} \tag{3.5a}$$

and compute

$$f_I = M_{Ik} + M_{IJ} f_J \, . \tag{3.5b}$$

In either case, if $f_k = 0$ go to Step 6. Otherwise continue.

<u>Step 5</u> (1 x 1 diagonal pivot) Set

$$J_{new} = \begin{cases} J_{old} \cup \{k\} & \text{if } k \notin J_{old} \\ J_{old} \setminus \{k\} & \text{otherwise.} \end{cases}$$

Go to Step 1.

<u>Step 6</u> (2 x 2 block pivot) If the vector $f = \begin{pmatrix} f_I \\ f_J \end{pmatrix}$ is nonnegative, stop, the complementarity problem (2.3) is infeasible. Otherwise determine another index ℓ by

$$-(\bar{q}_\ell + \theta \bar{p}_\ell) / f_\ell = \min \{ -(\bar{q}_i + \theta \bar{p}_i) / f_i \;\; : \;\; f_i < 0 \} \tag{3.6}$$

with θ being computed in (3.3). Set

$$J_{new} = \begin{cases} J_{old} \setminus \{k, \ell\} & \text{if } k, \ell \in J_{old} \\ J_{old} \setminus \{k\} \cup \{\ell\} & \text{if } k \in J_{old} \text{ and } \ell \notin J_{old} \\ J_{old} \setminus \{\ell\} \cup \{k\} & \text{if } \ell \in J_{old} \text{ and } k \notin J_{old} \\ J_{old} \cup \{k, \ell\} & \text{otherwise .} \end{cases}$$

Go to Step 1.

In the description of the algorithm above, J is the index set of the basic z-variables (cf. (2.3)), θ is the nonnegative parameter to be driven to zero and p is any vector satisfying the condition: $q + \theta^* p \geqq 0$ for some $\theta^* \geqq 0$.

Rigorously speaking, cycling could occur in the algorithm. Often this can be prevented by a lexicographic or least-index rule [1].

According to a basic result in pivotal algebra, the "basis matrix" M_{JJ} is nonsingular throughout the algorithm. Referring to the linear complementarity problem (2.2), the nonsingularity of M_{JJ} implies, among other things, that the numbers of basic η- and ξ-variables are bounded by $n+1$ and n respectively. (Recall that n is the order of the matrix A.)

4. THE COLUMN GENERATION TECHNIQUE

In this section, we show how the parametric principal pivoting algorithm described in the last section can be operated if the matrices P and Q are known only implicitly.

As pointed out at the end of the last section, at each iteration, there is a reasonable limit of information pertaining to the basic components. Consequently they can be stored without difficulty. However, it is necessary to derive an alternate way to handle the nonbasic components. From the description of the algorithm, it is obvious that these components are needed for the determination of the indices k and ℓ and the corresponding columns M_{Jk} and $M_{J\ell}$. As a matter of fact, the latter indices and columns are the key substances that one wishes to obtain at the end of the iteration. Consequently, if they can be determined without the complete knowledge of the nonbasic components, then one can operate the algorithm readily. In what follows, we demonstrate how this important step can be accomplished by means of a column generation technique.

Before starting, we would like to say a few words about the choice of the parametric vector p. In principle, it can be any vector satisfying the condition that $q + \theta^* p$ is nonnegative for some nonnegative θ^*. The latter condition ensures a valid start of the algorithm. However, in the present situation where the vector q is known only implicitly, the choice of p turns out to be quite crucial. This will become obvious in the analysis below.

4.1. The Bounded Case. We divide our discussion into two cases depending on whether the set X is bounded. We first consider the case where the X is bounded. In this case, the linear complementarity problem (2.2) reduces to

$$(4.1)\qquad \begin{aligned} &\varphi = P^T b + P^T A P \eta - \lambda^+ e + \lambda^- e \geq 0 && \eta \geq 0 \\ &\mu^+ = -1 + e^T \eta \qquad\qquad\qquad \geq 0 && \lambda^+ \geq 0 \\ &\mu^- = 1 - e^T \eta \qquad\qquad\qquad\; \geq 0 && \lambda^- \geq 0 \\ &\varphi^T \eta = (\lambda^+)^T \mu^+ = (\lambda^-)^T \mu^- = 0 \;. \end{aligned}$$

For the parametric vector, we choose $p = (e^T, 1, 0)^T$.

Recall that the objective for the ratio test in Step 3 is to determine a new critical value of the parameter θ such that vector $\bar{q} + \theta \bar{p}$ (whose components give the updated values of the currently basic variables) will remain nonnegative. Let K denote the index set of the currently basic η-variables. To keep the notations simple, we assume that both λ^+ and λ^- are nonbasic. (Notice that λ^+ and λ^- cannot be simultaneously basic.) The basis matrix is

$$M_{KK} = (P_K)^T A P_K$$

where P_K denotes the columns of P indexed by K. According to (3.1b), the nonbasic components are given by

$$(\bar{q}_I, \bar{p}_I) = \begin{pmatrix} (P_{\bar{K}})^T b & e_{\bar{K}} \\ -1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} (P_{\bar{K}})^T A P_K \\ e_K^T \\ -e_K^T \end{pmatrix} (\bar{q}_K \; \bar{p}_K)$$

where $\bar{K}$ denotes the complement of K and $P_{\bar{K}}$ the columns of P indexed by $\bar{K}$. Hence, $(\bar{q} + \theta\bar{p})_I$ is nonnegative if and only if

(4.2a) $$(P_{\bar{K}})^T[b + A P_K \bar{q}_K + \theta A P_K \bar{p}_K] \geq -\theta e_{\bar{K}}$$

(4.2b) $$-1 + e_K^T \bar{q}_K + \theta\ (1+e_K^T \bar{p}_K) \geq 0$$

(4.2c) $$1 - e_K^T \bar{q}_K - \theta\ e_K^T \bar{p}_K \geq 0 \quad .$$

Among these three conditions only the first one (4.2a) requires special attention because it involves the (unknown) matrix $P_{\bar{K}}$. Obviously, it follows from (3.1a) that for all values of θ,

(4.3) $$(P_K)^T [b + A P_K \bar{q}_K + \theta A P_K \bar{p}_K] = -\theta e_K \quad .$$

Thus (4.2a) holds if and only if

$$(P)^T [b + A P_K \bar{q}_K + \theta A P_K \bar{p}_K] \geq -\theta e$$

or equivalently

$$x^T [b + A P_K \bar{q}_K + \theta A P_K \bar{p}_K] \geq -\theta \qquad \text{for all } x \in X \quad .$$

Observe how the parametric vector is being used to derive this last inequality.

Consequently to determine the first value of θ for which (4.2a) is violated, one may proceed as follows: Solve the parametric linear program

(4.4) $$\text{minimize } L^1(\theta) = x^T[b + A P_K \bar{q}_K + \theta A P_K \bar{p}_K] \quad \text{subject to } x \in X$$

starting with the last critical value of θ for which the inequality below is satisfied

(4.5) $$L^1(\theta) + \theta \geq 0$$

and decreasing the parameter θ until either it reaches below zero or a value θ^1 - and a corresponding vector P_{k_1} - is obtained for which the inequality (4.5) is violated. For notational convenience, we let θ^1 be zero if θ reaches zero before the violation of the inequality. Notice that the vector P_{k_1} is an optimal solution to the linear program (4.4) with $\theta = \theta^1$. It is important to point out that if θ^1 is positive, the index k_1 obtained must be in $\bar{K}$ because equality in (4.3) holds for all values of θ.

With the value θ^1 available, the ratio test (3.3) can now be implemented as follows: Determine

$$\theta^2 = \max\{-\bar{q}_j/\bar{p}_j \quad : \quad \bar{p}_j > 0, \ j \in K\}$$

$$\theta^3 = \begin{cases} (1 - e_K^T \bar{q}_K)/e_K^T \bar{p}_K & \text{if } e_K^T \bar{p}_K < 0 \\ -(-1 + e_K^T \bar{q}_K)/(1 + e_K^T \bar{p}_K) & \text{if } -1 < e_K^T \bar{p}_K \leq 0 \\ -\infty & \text{otherwise} \end{cases}$$

and then set

$$\theta = \max\{\theta^1, \theta^2, \theta^3\} \quad . \tag{4.6}$$

This final value θ would give the desired new critical value. The maximizing index k can be determined in an obvious manner.

Essentially, the same analysis can be applied if either λ^+ or λ^- is basic. One may still implement the ratio test (3.3) by solving the same parametric linear program and by comparing the ratios among the basic components. A noteworthy point is that the inequality to be checked in this instance would be slightly different from (4.5), however.

Having performed the first ratio test, we proceed to show how the second one (3.6) can be carried out without the explicit knowledge of the whole matrix P. Before doing this, we point out that the column M_{Jk} required in (3.5a) is easy to generate once the maximizing index k is determined.

Recall that the ratio test in (3.6) is to determine the largest value $\rho \geq 0$ so that the vector $(\bar{q} + \theta \bar{p}) + \rho f$ remains nonnegative. Here ρ denotes that variable corresponding to the index k. To be specific we assume that k is equal to the index k_1 introduced earlier. Then, by (3.1b) and (3.5b) we have

$$(\bar{q} + \theta \bar{p})_I + \rho f_I = \begin{pmatrix} (P_{\bar{K}})^T[b + AP_K\bar{q}_K + \theta AP_K\bar{p}_K + \rho(AP_k + AP_K f_K)] + \theta e_{\bar{K}} \\ -1 + e_K^T\bar{q}_K + \theta(1 + e_K^T \bar{p}_K) + \rho(1 + e_K^T f_K) \\ 1 - e_K^T\bar{q}_K - \theta e_K^T \bar{p}_K - \rho(1 + e_K^T f_K) \end{pmatrix}$$

By using a similar analysis and by noting that the equality below is valid for all values of ρ, namely

$$(P_K)^T[b + A P_K \bar{q}_K + \theta A P_K \bar{p}_K + \rho(AP_k + AP_K f_K)] + \theta e_K = 0 \quad ,$$

one may easily obtain the minimum ratio (3.6) in the following manner: Solve the parametric linear program

(4.7) minimize $L^2(\rho) = x^T[b + AP_K\bar{q}_K + \theta AP_K\bar{p}_K + \rho(AP_k + AP_K f_K)]$ subject to $x \in X$

starting with the value $\rho = 0$ for which the inequality below is satisfied

(4.8) $L^2(\rho) + \theta \geq 0$

and increasing the parameter ρ until (4.8) is violated. Let ρ^1 be the first such value obtained and let P_{ℓ_1} be the corresponding optimal solution vector. If (4.8) remains satisfied for all values of ρ, set $\rho^1 = \infty$. Next, determine

$$\rho^2 = \min\{-(\bar{q}_i + \theta\bar{p}_i)/f_i \ : \ f_i < 0 \qquad i \in K\}$$

$$\rho^3 = \begin{cases} -[-1 + e_K^T\bar{q}_K + \theta(1 + e_K^T\bar{p}_K)]/(1 + e_K^T f_K) & \text{if } 1 + e_K^T f_K < 0 \\ -[1 - e_K^T\bar{q}_K - \theta e_K^T\bar{p}_K]/(1 + e_K^T f_K) & \text{if } 1 + e_K^T f_K > 0 \\ \infty & \text{otherwise} \end{cases}$$

and then set

$$\rho = \min\{\rho^1, \rho^2, \rho^3\} \quad .$$

The desired index ℓ can now be determined in an obvious manner.

Again, the same analysis can be applied if the maximum in (4.6) occurs at θ^3, i.e., if either λ^+ or λ^- is becoming basic. In this case, the parametric linear program (4.7) and the inequality (4.8) would be slightly changed but the essential idea would remain unchanged.

We point out that if K is the final index set of basic η-variables obtained at termination of the algorithm, then a solution to the stationary point problem is given by $u = P_K\eta_K$, which can be computed easily.

4.2 The Positive Definite Case. A considerably simplified version of the parametric principal pivoting algorithm can be used if the set X is bounded and if the matrix A is positive definite. The simplification stems from the fact that the linear complementarity problem (4.1) has an obvious reformulation under which the algorithm will never perform the 2 x 2 pivots. In particular, only one parametric linear program needs to be solved.

It is obvious that the linear complementarity problem (4.1) is equivalent to the following one: find λ and η such that

$$\varpi = P^Tb + \lambda e + P^TAP\eta \geq 0 \qquad \eta \geq 0 \qquad \varpi^T\eta = 0 \tag{4.9a}$$

$$1 = e^T\eta \qquad \text{and } \lambda \text{ unrestricted.} \tag{4.9b}$$

Condition (4.9a) constitutes a parametric linear complementarity problem (with λ as the parameter) to which the parametric principal pivoting algorithm is applicable. Concerning this application, we have the following result whose proof can be found in the reference cited.

Theorem 1. (Kaneko and Pang [5]) Suppose that the matrix A is positive definite. Consider the application of the parametric principal pivoting algorithm to the parametric linear complementarity problem (4.9a). Then each diagonal pivot entry f_k (cf. Step 4 of the algorithm) is positive and thus the 2 x 2 block pivots are redundant. In particular, the algorithm will always compute a solution to the problem for all values of λ. Let $\eta^*(\lambda)$ denote the solution obtained as a function of λ.

It is important to point out that for each λ, $\eta^*(\lambda)$ may not be the only solution to the linear complementarity problem (4.9a). In the rest of this subsection, we assume that A is positive definite.

Taking into consideration the condition (4.9b), we have

Theorem 2. Suppose that problem (4.9) has a solution $(\bar{\eta}, \bar{\lambda})$ with $\bar{\lambda} \neq 0$. Then each solution η^* to the linear complementarity problem (4.9a) with $\lambda = \bar{\lambda}$ also satisfies (4.9b). In particular, so does the solution $\pi^*(\bar{\lambda})$ obtained in Theorem 1.

<u>Proof</u>. Subtracting the two equations

$$\omega^* = P^T b + \bar{\lambda} e + P^T A P \pi^* \quad , \quad \bar{\omega} = P^T b + \bar{\lambda} e + P^T A P \bar{\pi}$$

we obtain

$$\omega^* - \bar{\omega} = P^T A P(\pi^* - \bar{\pi})$$

which implies

$$0 \geqq (\pi^* - \bar{\pi})^T (\omega^* - \bar{\omega}) = (P(\pi^* - \bar{\pi}))^T AP(\pi^* - \bar{\pi}) \geqq 0 \ .$$

By the positive definiteness of A, it follows that

$$P(\pi^* - \bar{\pi}) = 0 \ .$$

Consequently, we have

$$0 = (\pi^*)^T \omega^* = (P\pi^*)^T b + \bar{\lambda}\, e^T \pi^* + (P\pi^*)^T AP\pi^*$$

$$= \bar{\lambda}\, (e^T \pi^* - e^T \bar{\pi}) = \bar{\lambda}(e^T \pi^* - 1) \ .$$

By assumption, $\bar{\lambda} \neq 0$; hence the desired conclusion follows. Q.E.D.

Now, if the parametric principal pivoting algorithm is applied to the parametric linear complementarity problem (4.9a), then either a value $\bar{\lambda}$ can be found for which the corresponding solution $\pi^*(\bar{\lambda})$ satisfies the condition (4.9b) as well, or no such value exists. In the first case, a solution to the stationary point problem is obtained. In the second case, there are two possibilities: either the problem (4.9) has a solution with $\lambda = 0$ or it has no solution at all. To determine this, we may solve the system of linear inequalities:

$$(4.10) \qquad P\pi = P\pi^*(0) \quad , \quad e^T\pi = 1 \quad \text{and} \quad \pi \geqq 0.$$

<u>Theorem 3</u>. Suppose that the above version of the parametric principal pivoting algorithm does not compute a solution to the problem (4.9). If the system (4.10) is inconsistent, then problem (4.9)(and therefore the stationary point problem) has no solution. On the other hand, if $\bar{\pi}$ is any solution to (4.10), then $(\bar{\pi}, 0)$ solves (4.9).

<u>Proof</u>. By Theorem 2 and the assumption, it follows that any solution *(if it exists) to problem (4.9) must have $\lambda = 0$.* Suppose that (4.9) does have a solution $(\bar{\pi}, 0)$. We show $P\bar{\pi} = P\pi^*(0)$. This follows from a more general result whose proof is easy and omitted: If π^1 and π^2 are two solutions to the linear complementarity problem (4.9a) for the same value of λ, then $P\pi^1 = P\pi^2$. (Cf. the proof of the last theorem.) Consequently, if the system (4.10) is inconsistent, then problem (4.9) has no solution.

Conversely, let $\bar{\pi}$ be any solution to the system (4.10). It suffices to show that $(\bar{\pi}, 0)$ satisfies (4.9a). Obviously,

$$P^Tb + P^TAP\bar{\pi} = P^Tb + P^TAP\pi^*(0) \geqq 0$$

and

$$\bar{\pi}^T(P^Tb + P^TAP\bar{\pi}) = (\pi^*(0))^T (P^Tb + P^TAP\pi^*(0)) = 0 .$$

This completes the proof of the theorem. Q.E.D.

In what follows, we outline how the ideas presented above can be actually carried out to solve the stationary point problem in the case where the matrix P is known only implicitly. First, by the column generation technique described in the last subsection, the parametric principal pivoting algorithm applied to (4.9a) can in fact be implemented. The search for the suitable $\bar{\lambda}$ can be achieved by a simple interpolation procedure described in [7]. The procedure does not involve the matrix P and is therefore applicable. Finally, the consistency of the system (4.10) can be determined by checking if the vector $P\pi^*(0)$ is in the set X. Notice that this vector is equal to $P_K[\pi^*(0)]_K$ where K is the index set of basic π-variables. Since P_K is known, $P\pi^*(0)$ can be obtained. Notice also that a solution π to the system (4.10) is not actually needed to obtain a solution to the stationary point problem. This is because if the vector $P\pi^*(0)$ is in X, then it is a desired solution.

4.3 The Unbounded Case. We extend the discussion of Section 4.1 to treat the case where the set X is not necessarily bounded.

Associated with the set X is its "homogenized" cone

$$Y = \{ \binom{x}{t} \in R^{n+1} : Cx \geqq ct \quad , t \geqq 0 \} .$$

We assume that the dual cone Y* contains an interior point which is available explicitly.[2/] Such a point $\binom{y^*}{s^*}$ will be used to define the parametric vector p and is characterized by the condition: $(y^*)^T x + s^* t > 0$ for all $\binom{x}{t} \in Y \setminus \{0\}$. In particular, it holds that

$$(y^*)^T P + s^* e > 0 \qquad \text{and} \qquad (y^*)^T Q > 0 .$$

2/ We shall discuss more about this assumption in the Appendix.

Choose

$$p = ((y^*)^T P + s^* e^T \quad , \; (y^*)^T Q \quad , \quad 1 \quad , \quad 0)^T \; .$$

The reason for this choice will become obvious in a moment. We should point out that it is not necessary to compute the vector p explicitly. In fact, its components will be generated when they are needed.

Following the analysis of Section 4.1, we consider a typical iteration of the parametric principal pivoting algorithm applied to the linear complementarity problem (2.2) with the parametric vector p chosen above. Let K and L denote respectively, the index sets of the currently basic η- and ξ-variables. This time, we assume that λ^+ is also basic. The basis matrix is then

$$M_{JJ} = \begin{pmatrix} (P_K)^T A P_K & (P_K)^T A Q_L & -e_K \\ (Q_L)^T A P_K & (Q_L)^T A Q_L & 0 \\ e_K^T & 0 & 0 \end{pmatrix}$$

Partitioning the basic components in accordance with the basis matrix, we may write

$$(\bar{q}_J \; , \; \bar{p}_J) = \begin{pmatrix} \bar{q}_K^1 & \bar{p}_K^1 \\ \bar{q}_L^2 & \bar{p}_L^2 \\ t^1 & t^2 \end{pmatrix}$$

According to (3.1b), the nonbasic components are given by

$$(\bar{q}_I, \bar{p}_I) = \begin{pmatrix} (P_{\bar{K}})^T b & (P_{\bar{K}})^T y^* + s^* e_{\bar{K}} \\ (Q_{\bar{L}})^T b & (Q_{\bar{L}})^T y^* \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} (P_K c)^T A P_K & (P_{\bar{K}})^T A P_K & -e_{\bar{K}} \\ (Q_{\bar{L}})^T A P_K & (Q_{\bar{L}})^T A Q_L & 0 \\ -e_K^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \bar{q}_K^1 & \bar{p}_K^1 \\ \bar{q}_L^2 & \bar{p}_L^2 \\ t^1 & t^2 \end{pmatrix}$$

Hence $(\bar{q} + \theta\bar{p})_I$ is nonnegative if and only if

(4.11a) $(P_{\bar{K}})^T\{b + A(P_K\bar{q}_K^1 + Q_L\bar{q}_L^2) + \theta[y^* + A(P_K\bar{p}_K^1 + Q_L\bar{p}_L^2)]\} \geqq [t^1 + \theta(t^2 - s^*)]e_{\bar{K}}$

(4.11b) $(Q_{\bar{L}})^T\{b + A(P_K\bar{q}_k^1 + Q_L\bar{q}_L^2) + \theta[y^* + A(P_K\bar{p}_K^1 + Q_L\bar{p}_L^2)]\} \geqq 0$

(4.11c) $(1 - e_K^T\bar{q}_K^1) - \theta e_K^T\bar{p}_K^1 = \theta \geqq 0$

Obviously, equality holds for all values θ in (4.11a) and (4.11b) if $\bar{K}$ and $\bar{L}$ are replaced by K and L respectively. Consequently, (4.11a) and (4.11b) hold if and only if

$$x^T\{b + A(P_K\bar{q}_K^1 + Q_L\bar{q}_L^2) + \theta[y^* + A(P_K\bar{p}_K^1 + Q_L\bar{p}_L^2)]\} \geqq t^1 + \theta(t^2 - s^*) \text{ for all } x \in X .$$

Therefore, to determine the first value of θ for which the condition (4.11) is violated, one may proceed as follows: Solve the parametric linear program

$$\text{minimize } L^3(\theta) = x^T\{b + A(P_K\bar{q}_K^1 + Q_L\bar{q}_L^2) + \theta[y^* + A(P_K\bar{p}_K^1 + Q_L\bar{p}_L^2)]\} \text{ subject to } x \in X$$

starting with the last critical value of θ for which the inequality below is satisfied

(4.12) $L^3(\theta) \geqq t^1 + \theta(t^2 - s^*)$

and decreasing θ until either it reaches below zero or a vlaue θ^1 is obtained for which the inequality (4.12) is violated. In the latter case, either an index $k_1 \in \bar{K}$ (and a corresponding optimal solution vector P_{k_1}) or an index $k_1' \in \bar{L}$ (and a corresponding ray vector $Q_{k_1'}$) is obtained as well. For convenience, we let θ^1 be zero if (4.12) remains valid at $\theta = 0$.

With the value θ^1 determined the ratio test (3.3) can now be carried out by comparing θ^1 with the maximum ratio among the basic components.

By a similar analysis, it can be shown that the second ratio test (3.6) can be implemented by solving a parametric linear program like the one (4.7) and by checking when a certain inequality involving the optimum objective value and the parameter of the program (cf. (4.8)) is violated. The details are omitted.

Finally, we point out that if K and L are the final index sets of basic π- and ξ-variables, then a stationary point is given by $u = P_K \pi_K + Q_L \xi_L$.

5. CONCLUSION

In this paper, we have shown how the parametric principal pivoting algorithm can be used to compute a stationary point of an affine function over a polyhedral set. The success of this approach depends very much on a proper choice of the parametric vector which allows the algorithm to operate solely with the basic ingredients. At this moment, there seems to be a theoretical drawback, however. Namely, we have not yet been able to settle the question of cycling in the implementation. One possible way to do this would be to extend the well-known lexicographic rule (see [2] e.g). The key issue here is the question of how to incorporate the rule properly in the linear complementarity problem (2.2). Special care is needed because the form of the constant vector plays a very crucial role in the column generation technique.

Fortunately, the devil of cycling has always been (and hopefully, will be) a theoretical concern mainly. So, even without a theoretically justified cycling-prevention scheme, it would seem that the technique proposed should still deserve a try for solving practical applications.

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APPENDIX

We establish

Theorem 4. Let S be a polyhedral cone with the representations

$$S = \{x : Cx \geq 0\} = \{x : x = P\lambda \text{ for some } \lambda \geq 0\} .$$

where each generator P_i is nonzero. Then the following are equivalent

(i) The vector y^* is an interior point of the dual cone S^*

(ii) $(y^*)^T x > 0$ for all $x \in S \setminus \{0\}$

(iii) $(y^*)^T P > 0$

(iv) For each vector p, there exist a vector $\lambda \geq 0$ and a scalar $\mu \geq 0$ such that

$$C^T \lambda = p + \mu y^*$$

(v) For each vector p, there exists a scalar $\mu \geq 0$ such that

$$0 = \min_{x \in S} (p + \mu y^*)^T x .$$

Proof. The equivalence of (i), (ii), and (iii) are well-known. That of (iv) and (v) is an immediate consequence of the duality theory of linear programming. We now show that (ii) is equivalent to (iv). Obviously, (ii) is equivalent to the fact that for all vectors p, the system below has no solution

$$Cx \geq 0 \quad , \quad -(y^*)^T x \geq 0 \quad \text{and} \quad p^T x < 0$$

By Motzkin's theorem of the alternative, the latter fact is in turn equivalent to that for each vector p, there exist a $\lambda \geqq 0$ and $\mu \geqq 0$ such that

$$-p + C^T\lambda - \mu y^* = 0$$

which is precisely condition (iv). Q. E. D.

If a polyhedral cone is represented in terms of its generators, then according to (iii), the task of finding an interior point of the dual cone can be accomplished by solving a linear program. On the other hand, if the primal cone is given in terms of a system of linear inequalities, then it may not be easy to determine whether such a point exists. However, there are cases where this is trivial. For instance, if the given cone S lies in the nonnegative orthant, then the vector of ones is obviously a desired interior point of S* by (ii).

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In two recent papers, Eaves showed that Lemke's algorithm can be used to compute a stationary point of an affine function over a polyhedral set. This paper proposes an alternative method which is based on parametric principal pivoting. The proposed method involves solving systems of linear equations and parametric linear subprograms over the given polyhedral set. An obvious advantage of the method is that any special structure of the polyhedral set can be exploited profitably in the solution of the subprograms.

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